Complexity of Tensor Rank Computations

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1 Rank Of Boolean Matrices

In this section, we devote our attention to *Boolean* matrices. These matrices are over the *Boolean semi-ring* $< \{0,1\}, +, \cdot >$ in which the elements 0 and 1 are added and multiplied exactly like integers, with the exception that $1+1=1$. The sum and product of Boolean matrices is defined as usual, but the addition and multiplication of elements are, of course, operations over the Boolean semi-ring.

The *rank* of an $m \times n$ Boolean matrix M is defined to be the least integer r for which there exist Boolean matrices A and B of orders $m \times r$ and $r \times n$, respectively, such that $M = AB$. Since M can always be written down as a matrix product with the identity matrix of order $m \times m$ (resp. $n \times n$) as the left (resp. right) factor—the other factor being M itself— the rank of M is defined and is at most $\min(m, n)$.

We want to show that the problem of rank determination is NP-complete for Boolean matrices. For this, it is clearly sufficient to prove the NP-completeness of the following related problem.

BOOLEAN MATRIX FACTORIZATION

- Instance: An $m \times n$ Boolean matrix M, positive integer $r \leq n$.
- Question: Are there Boolean matrices A and B of orders $m \times r$ and $r \times n$, respectively, such that $M = AB$?

The following problem is known to be NP-complete (Problem SP7 on p. 222 in [1]):

SET BASIS

- Instance: Collection C of subsets of a finite set S, positive integer $g \leq |C|$.

- Question: Is there a collection D of subsets of S, with $|D| = g$, such that, for each $c \in C$, there is a subcollection of D whose union is exactly c?

BOOLEAN MATRIX FACTORIZATION is in NP because one can guess the Boolean matrices A and B and then verify in polynomial time that $M = AB$. We prove that BOOLEAN MATRIX FACTORIZATION is NP-complete by transforming SET BASIS to it.

Suppose an arbitrary instance of SET BASIS consists of a set $S = \{s_1, \ldots, s_n\}$, a collection $C = \{c_1, \ldots, c_m\}$ with each $c_i \subseteq S$, and a positive integer $g \leq m$. We construct an instance of BOOLEAN MATRIX FACTORIZATION as follows: Let M be an $m \times n$ matrix whose generic element m_{ij}, for $1 \leq i \leq m, 1 \leq j \leq n$, is given by

$$m_{ij} = \begin{cases} 1, & \text{if } s_j \in c_i; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $r = g$.

To complete the problem transformation, we need to show that a collection D, with $|D| = g = r$, exists for SET BASIS if and only if there are Boolean matrices A and B of orders $m \times r$ and $r \times n$, respectively, such that $M = AB$. The proof, which is omitted, is a simple consequence of the following definition of A and B in terms of D. Suppose D exists and can be written as $\{d_1, \ldots, d_r\}$. For each $1 \leq i \leq m$, c_i is equal to the union of a subcollection of D. Let A and B be matrices whose generic elements a_{ik} and b_{kj}, for $1 \leq i \leq m, 1 \leq k \leq r, 1 \leq j \leq n$, are given by

$$a_{ik} = \begin{cases} 1, & \text{if } d_k \subseteq c_i; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$b_{kj} = \begin{cases} 1, & \text{if } s_j \in d_k; \\ 0, & \text{otherwise.} \end{cases}$$

The above problem transformation is illustrated as follows. Let an instance of SET BASIS be given by: $S = \{s_1, s_2, s_3\}, C = \{\{s_1, s_3\}, \{s_2\}, \{s_1, s_2, s_3\}\}, g = 2$. For this instance, the answer to the problem is YES, with the subcollection D being $\{\{s_1, s_3\}, \{s_2\}\}$. The corresponding instance of BOOLEAN MATRIX FACTORIZATION is given by

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad r = 2.$$

The answer to this problem is again YES, with $M = AB$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

References

[1] Garey, M.R. & Johnson, D.S.: *Computers and Intractability: A Guide to the Theory of NP-Complteness*, W.H. Freeman, New York, 1979.